An Extension of the Wigner-Araki-Yanase Theorem to Multiplicative Conserved Quantities

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An extension of the Wigner-Araki-Yanase theorem to multiplicative conserved quantities is presented and approximate versions of the theorem are discussed. The theorem proposed by Wigner, and subsequently proven by Araki and Yanase in a general setting, deals with the impossibility of exact nondestructive measurements of observables that do not commute with additive conserved quantities. An analogous theorem concerning the limitation of nondestructive measurements of multiplicative conserved quantities is proven in this paper. The result is analyzed in the context of earlier work. An approximate form of the theorem, more appropriate in experimental settings, as well as possible extensions are also briefly discussed.

I. INTRODUCTION

An extension of the Wigner-Araki-Yanase theorem is presented and an approximate version of the theorem is discussed. The theorem proposed by Wigner [1] following an earlier paper by Lüders[2], and subsequently proven by Araki & Yanase [3], deals with the impossibility of exact nondestructive quantum mechanical measurements of observables that do not commute with additive conserved quantities. An analogous theorem concerning the limitation of nondestructive measurements of multiplicative conserved quantities that are related to discrete symmetries is proven here. The result is analyzed in the context of earlier work. An approximate form of the theorem, more appropriate in experimental settings, is also briefly analyzed.

The intense study of quantum computers has led to a revival of interest in foundational issues in quantum mechanics including various aspects of the measurement process. One result in this context is the further analysis of approximate versions of the Winger-Arkai-Yanase theorem. This work has been carried out in particular by Ozawa [4, 5], who has discussed implications of the theorem for the construction of quantum gates.

Let us restate the Wigner-Araki-Yanase theorem briefly. An additive observable acting on the combined Hilbert space $H_1 \otimes H_2$, where H_1 and H_2 are the Hilbert spaces associated with the measured object and the measurement apparatus respectively, has the following form $\hat{L}^A \otimes \hat{1} + \hat{1} \otimes \hat{L}^B$, where $\hat{L}^A$ is an observable[8] of the observed system and $\hat{L}^B$ describes the measuring apparatus. This additive observables is a *conserved* quantity, which is generally associated with a continuous symmetry, if it satisfies the following equation

$$\hat{L}^A \otimes \hat{1} + \hat{1} \otimes \hat{L}^B = \hat{U}^\dagger(\hat{L}^A \otimes \hat{1} + \hat{1} \otimes \hat{L}^B)\hat{U}, \tag{1}$$

where $\hat{U}$ is the evolution operator of the combined system. The previous equation (1) is equivalent to $[\hat{U}, \hat{L}^A \otimes \hat{1} + \hat{1} \otimes \hat{L}^B] = 0$. The Wigner-Araki-Yanase theorem now states under some reasonable assumptions about the observable and the Hilbert spaces that one can only measure an observable $\hat{O} \otimes \hat{1}$ of the system in a nondestructive way, if $[\hat{O}, \hat{L}^A] = 0$. For clarification, a nondestructive measurement is defined to be a measurement, which leaves the system in its original eigenstate. As a further clarification, this does not violate the non-cloning theorem, since the measurement is only nondestructive for the preselected basis of the original Hilbert space H_1 and not for any superposition of basis elements.

In the following section the Wigner-Araki-Yanase theorem is extended in a novel way to multiplicative conserved quantities. A description of an approximate version of the theorem is presented in a subsequent section, which leads to an inequality similar in nature, but different in form, to the one already established by Ozawa. The paper is rounded off by a brief speculation about further extensions of the theorem.

II. THE WIGNER-ARAKI-YANASE THEOREM FOR MULTIPLICATIVE CONSERVED QUANTITIES

In the following paragraphs a theorem for multiplicative conserved quantities is proven that extends the Wigner-Araki-Yanase theorem. The notation introduced above is employed without modification wherever possible. A multiplicative quantity can be written as $\hat{L}^A \otimes \hat{L}^B$. It is a multiplicative *conserved* quantity, if

$$\hat{L}^A \otimes \hat{L}^B = \hat{U}^\dagger(\hat{L}^A \otimes \hat{L}^B)\hat{U}. \tag{2}$$

Now we can proof under some reasonable assumptions - i.e both Hilbert spaces are finite dimensional with H_1 n_1-dimensional and H_2 less than $2n_1$-dimensional, the rank of $\hat{L}^B$ is maximal, and all the eigenvalues of $\hat{L}^A$ and $\hat{L}^B$ are positive- that every observable $\hat{O}$ that does not satisfies $[\hat{O}, \hat{L}^A] = 0$ cannot be measured exactly in a nondestructive way. Let us next state the theorem precisely and then give the proof.

Theorem: If an operator $\hat{O}$ can be measured exactly in a nondestructive way, then it has to satisfy the property $[\hat{O}, \hat{L}^A] = 0$ under the assumption that $\hat{L}^A \otimes \hat{L}^B$ is a multiplicative conserved quantity, i.e. $U^\dagger(\hat{L}^A \otimes \hat{L}^B)U = \hat{L}^A \otimes \hat{L}^B$, $\hat{L}^B$ has maximal rank, the Hilbert spaces of the system H_1 is n_1-dimensional, and the Hilbert space of the measurement apparatus H_2 is less than $2n_1$-dimensional. We further assume that all eigenvalues of $\hat{L}^A$ and $\hat{L}^B$ are greater than zero.

Proof: Consider $\langle u(i)|\hat{L}^A|u(j)\rangle\langle v|\hat{L}^B|v\rangle$ for any i & j between 1 and n_1, which transforms to

$$\begin{aligned} & \langle u(i) \otimes v|\hat{L}^A \otimes \hat{L}^B|u(j) \otimes v\rangle & (3)\\ = \; & \langle u(i) \otimes v|\hat{U}^\dagger(\hat{L}^A \otimes \hat{L}^B)\hat{U}|u(j) \otimes v\rangle & (4)\\ = \; & \langle u(i)|\hat{L}^A|u(j)\rangle\langle v(i)|\hat{L}^B|v(j)\rangle, & (5)\end{aligned}$$

since $\hat{L}^A \otimes \hat{L}^B$ is a conserved quantity. The equality

$$\langle u(i)|\hat{L}^A|u(j)\rangle\langle v|\hat{L}^B|v\rangle = \langle u(i)|\hat{L}^A|u(j)\rangle\langle v(i)|\hat{L}^B|v(j)\rangle \tag{6}$$

can either be satisfied, if $\langle v(i)|\hat{L}^A|v(j)\rangle = \text{const}$ or by $\langle u(i)|\hat{L}^A|u(j)\rangle = 0$. The second case immediately implies the theorem.Therefore, we only have to show that the first case leads to a contradiction. This can be shown in the following way. First we expand $\hat{L}^B|v(j)\rangle$ in terms of the basis for $|v\rangle$:

$$\hat{L}^B|v(j)\rangle = \sum_{k=1}^{n_2} L^B_{kj}|v(k)\rangle \tag{7}$$

Next we exploit the orthogonality of the basis for $|v\rangle$ to get: $\langle v(i)|\hat{L}^B|v(j)\rangle = L^B_{ij}$. From this it follows that the sub-matrix of L^B_{ij} with $i, j \in 1, ..., n_1$ is at most of rank one. The full matrix L^B_{ij} is of maximal rank, as stated in the assumptions, implying that the sub-matrix L^B_{ij} with $i \in 1, ..., n_1$ and $j \in n_1 + 1, ..., n_2$ is $n_1 - 1$-dimensional. This is only possible, if n_2 is at least $2n_1$, which contradicts one of the assumptions. Therefore, $\langle u(i)|\hat{L}^A|u(j)\rangle = 0$ for all $i, j \in 1, ..., n_1$, and like in the proof of the original Wigner-Araki-Yanase this implies $[\hat{O}, \hat{L}^A]$. **QED**.

Examples for multiplicative conserved quantities can be found in particle physics, where parity is a multiplicative conserved quantity, and other areas where discrete symmetries are relevant.

III. AN APPROXIMATE VERSION OF THE INEQUALITY

In this section we provide an approximate version of the theorem proven above in terms of an inequality. For practical purposes, i.e. to be able to point out theoretical limits in experiments, it is necessary to go beyond a no-go theorem and deal with approximate versions of the statement. Particular interest in areas like quantum information theory has been shown for measurements which are nondestructive but only distinguish the potential input states with less than 100% accuracy by correlating the measurement

apparatus imperfectly with the state of the system. One natural way, pioneered by Ozawa [5], is to introduce a *Noise operator* $\hat{N}$ defined as

$$\hat{N} := \hat{M}(t + \Delta t) - \hat{O}(t), \tag{8}$$

where $\hat{M}(t + \Delta t)$ is the potentially error-prone probe observable, i.e. $\langle v(i)|v(j)\rangle$ is not necessarily zero. One could, of course, pursue other approaches, e.g. for example assume that either the measurement is not exactly nondestructive or that the conserved quantity is not perfectly conserved. These other approaches will be discussed in a separate paper[6]. Here we follow closely the reasoning of Ozawa, who derived in the additive case an inequality for a quantity, $\epsilon(\psi) = \langle N^2\rangle$ associated with the state-dependent measurement noise, related to the noise operator. After some simple manipulations we can show that the measurement noise $\epsilon(\psi)$ is bounded below by a function of the various other operators involved. The inequality has the form

$$\epsilon(\psi)^2 \geq \frac{|\langle[\hat{O}(t), \hat{L}^A] \otimes \hat{L}^B - \hat{L}^A \otimes [\hat{M}(t + \Delta t), \hat{L}^B]\rangle|^2}{4(\Delta\hat{L}^A)^2(\Delta\hat{L}^B)^2}. \tag{9}$$

In the derivation of the inequality we utilized the Cauchy-Schwartz inequality, $\epsilon(\psi) \geq (\Delta N)^2$, and that $\Delta[\hat{L}^A \otimes \hat{L}^B]^2 = (\Delta\hat{L}^A\Delta\hat{L}^B)^2$. The last property follows from the fact that the variance of product states is the product of the variances of its components.

To simplify the inequality we assume the Yanase condition[7]: $[\hat{M}, \hat{L}^B] = 0$. The inequality then takes the form

$$\epsilon(\psi)^2 \geq \frac{|\langle[\hat{O}(t), \hat{L}^A] \otimes \hat{L}^B\rangle|^2}{4(\Delta\hat{L}^A)^2(\Delta\hat{L}^B)^2}. \tag{10}$$

If the expectation value of $\hat{L}^B$ is zero this further simplifies to

$$\epsilon(\psi)^2 \geq \frac{|\langle[\hat{O}(t), \hat{L}^A]\rangle|^2}{4(\Delta\hat{L}^A)^2}. \tag{11}$$

Unlike in the additive case in this very special multiplicative case with $\langle\hat{L}^B\rangle = 0$ a decrease in this particular form of the lower bound for the measuring noise cannot simply be achieved by increasing the variance of $\hat{L}^B$.

IV. EXTENSIONS

There are various extensions one ought to consider. One particular interesting case is to establish the Wigner-Araki-Yanase Theorem for generalized conserved quantities with both additive as well as multiplicative components. This extension is important, when one considers for example a generalized interaction between system and environment, and might be testable experimentally.

Also of interest would be to explore the relationship between multiplicative conserved quantities and the implementation limitations for quantum computers. Of particular interest is also to explore in general the relation between additive and multiplicative version of the Wigner-Araki-Yanase theorem. This and other extensions can be found in an enlarged version of the article, which is under preparation [6].

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